

# A CHARACTERIZATION OF BLD-MAPPINGS BETWEEN METRIC SPACES

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**ABSTRACT.** We prove a characterization theorem for BLD-mappings between complete locally compact path-metric spaces. As a corollary we obtain a sharp limit theorem for BLD-mappings.

## 1. INTRODUCTION

The class of BLD-mappings was introduced in [MV88] as mappings that preserve solutions of certain elliptic partial differential equations. In that paper Martio and Väisälä showed, among other results, that the class of BLD-mappings has several equivalent definitions. In this paper we use the following geometric definition. For the definitions of the length of a path, path-metric spaces and branched covers, see Section 2.

**Definition.** Given  $L \geq 1$ , a branched cover  $f: X \rightarrow Y$  between metric spaces is a mapping of *Bounded Length Distortion*, or  $(L)$ -BLD for short, if for all paths  $\gamma: [0, 1] \rightarrow X$ , we have

$$(BLD) \quad L^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq L\ell(\gamma).$$

In [MV88] Martio and Väisälä defined BLD-mappings as a subclass of the so called *quasiregular mappings*, see e.g. [MV88] or [Ric93], and showed that this analytic definition is equivalent to the geometric definition given above. There are, however, also other characterizations of BLD-mappings between Euclidean spaces in [MV88] although they are not explicitly stated as such. In this paper we state these characterizing properties and prove a characterization (Theorem 1.1), which shows that the equivalent definitions for BLD-mappings in [MV88] hold true also in the setting of complete locally compact path-metric spaces.

Note that the path-length condition (BLD) is required for *all* paths. This requirement implies that the lifts of rectifiable paths are also rectifiable. (For terminology, see Section 2.) Requiring the path-length condition (BLD) only for rectifiable paths gives rise to the class of *weak BLD-mappings*, that have been studied e.g. by Hajłasz and Malekzadeh, see [HM15b]. In Euclidean spaces the two definitions are equivalent (see [HM15a]) but, in general, weak BLD-mappings form a strictly larger class of mappings. For example the identity map  $\mathbb{H}_1 \rightarrow \mathbb{R}^3$  from the first Heisenberg group to the Euclidean 3-space is a weak BLD-mapping but not a BLD-mapping.

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A mapping  $f: X \rightarrow Y$  between metric spaces is  $L$ - $LQ$  (*Lipschitz Quotient*) if, for all  $x \in X$  and  $r > 0$ ,

$$(LQ) \quad B_Y(f(x), L^{-1}r) \subset f(B_X(x, r)) \subset B_Y(f(x), Lr).$$

$LQ$ -mappings were introduced with this name in [BJL<sup>+</sup>99]. Note, however, that Martio and Väisälä show already in [MV88, Lemma 4.6.] that  $BLD$ -mappings satisfy the  $(LQ)$  property locally in Euclidean domains. For mappings between complete and locally compact path-metric spaces the definition of  $L$ - $LQ$ -mappings is equivalent to a local one; see Lemma 2.3. The definition of  $LQ$ -mappings immediately yields that  $LQ$ -mappings are open, but they are not necessarily discrete; the projection map  $\mathbb{R}^2 \rightarrow \mathbb{R}$  is a trivial example, but see also [Csö01] for a construction of Csörnyei for an  $LQ$ -mapping  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  with a point  $P \in \mathbb{R}^2$  such that  $f^{-1}(\{P\})$  contains a plane.

Let  $f: X \rightarrow Y$  be a continuous mapping between path-connected metric spaces. As in [Ric93, II.4] and [HR02, p. 491] we set

$$\begin{aligned} L(x, f, r) &:= \sup\{d(f(x), f(y)) \mid y \in \partial B(x, r)\}, \\ l(x, f, r) &:= \inf\{d(f(x), f(y)) \mid y \in \partial B(x, r)\}, \\ L^*(x, f, r) &:= \sup\{d(x, y) \mid y \in \partial U(x, f, r)\}, \end{aligned}$$

and

$$l^*(x, f, r) := \inf\{d(x, y) \mid y \in \partial U(x, f, r)\},$$

where  $U(x, f, r)$  the component of  $f^{-1}(B(f(x), r))$  containing  $x$ . Note that since  $X$  is a path-metric space, for every  $x \in X$   $\partial U(x, f, r) \neq \emptyset$  for all  $r > 0$  small enough when  $f$  is not a constant map.

A mapping  $f: X \rightarrow Y$  is  $L$ -*radial* if for all  $x \in X$  there exists a radius  $r_0 > 0$  such that for all  $r < r_0$

$$(R) \quad L(x, f, r) \leq Lr \quad \text{and} \quad l(x, f, r) \geq L^{-1}r.$$

An equivalent definition for radial mappings is given in Lemma 2.4.

Likewise we say that a branched cover  $f: X \rightarrow Y$  is  $L$ -*coradial* if for all points  $x \in X$  there exists a radius  $r_0 > 0$  such that for all  $r < r_0$

$$(R^*) \quad L^*(x, f, r) \leq Lr \quad \text{and} \quad l^*(x, f, r) \geq L^{-1}r.$$

As branched covers coradial mappings are continuous, open and discrete by definition. For radial mappings the radially condition (R) immediately implies that an  $L$ -radial mapping is both discrete and locally  $L$ -Lipschitz. On the other hand radial maps are not necessarily open, as the example  $(x, y) \mapsto (|x|, y)$  defined in the Euclidean plane shows.

Our first main theorem is the following characterization.

**Theorem 1.1.** *Let  $f: X \rightarrow Y$  be a continuous mapping between two complete locally compact path-metric spaces and  $L \geq 1$ . Then the following are equivalent:*

- (i)  $f$  is an  $L$ - $BLD$ -mapping,
- (ii)  $f$  is a discrete  $L$ - $LQ$ -mapping,
- (iii)  $f$  is an open  $L$ -radial mapping, and
- (iv)  $f$  is an  $L$ -coradial mapping.

Theorem 1.1 generalizes and extends an earlier result by the author, see [Lui, Theorem 1.1]. As mentioned, this result is known in the Euclidean setting [MV88]. The implication  $(i) \Rightarrow (ii)$  is observed in the setting of generalized manifolds of type  $A$  in [HR02], and more recently the implication  $(i) \Rightarrow (iii)$  is noted in a setting similar to [HR02] under additional assumptions on spaces  $X$  and  $Y$  by Guo and Williams [GW]. The implication  $(iii) \Rightarrow (ii)$  is implicitly due to Lytchak in a purely metric setting without notions of branched covers, see [Lyt05, Section 3.1 and Proposition 4.3]. Furthermore in [HR02, Theorem 4.5] it is shown that a mapping between quasiconvex generalized manifolds is BLD if and only if it is locally regular in the sense of David and Semmes, see [DS97, Definition 12.1]. That equivalence does not, however, preserve the constant  $L$ .

Locally uniform limits of  $L$ -LQ mappings are  $L$ -LQ in a very general setting and so Theorem 1.1 yields that the  $L$ -BLD condition passes to limits of BLD-mapping packages (defined in Section 4) when the limiting map is discrete. More precisely, we have the following theorem.

**Theorem 1.2.** *Let  $(X_j, x_j)$  and  $(Y_j, y_j)$  be two pointed sequences of locally compact and complete path-metric spaces. Suppose the sequence of pointed mapping packages  $((X_j, x_j), (Y_j, y_j), f_j)$ , where each  $f_j: (X_j, x_j) \rightarrow (Y_j, y_j)$  is  $L$ -BLD, converges to a mapping package  $((X, x_0), (Y, y_0), f)$  where  $f$  is discrete. Then  $f$  is  $L$ -BLD.*

As an immediate corollary we get a result for fixed spaces.

**Corollary 1.3.** *Let  $X$  and  $Y$  be locally compact complete path-metric spaces and suppose  $(f_j)$  is a sequence of  $L$ -BLD-mappings  $X \rightarrow Y$  converging pointwise to a continuous discrete mapping  $f: X \rightarrow Y$ . Then  $f$  is  $L$ -BLD.*

In the setting of path-metric generalized manifolds of type  $A$ , we may deduce the discreteness of a limit of  $L$ -BLD mappings from uniform bounds for local multiplicity. Thus in this setting we have the following stronger result.

**Theorem 1.4.** *Let  $(M_j, x_j)$  and  $(N_j, y_j)$  be two pointed sequences of path-metric generalized  $n$ -manifolds of type  $A$  with uniform constants. Suppose the sequence of pointed mapping packages  $((M_j, x_j), (N_j, y_j), f_j)$  converges to a mapping package  $((M, x_0), (N, y_0), f)$ , where each  $f_j: M_j \rightarrow N_j$  is  $L$ -BLD. Then  $f$  is  $L$ -BLD.*

We again have a corresponding result for fixed spaces as a corollary.

**Corollary 1.5.** *Let  $M$  and  $N$  be generalized  $n$ -manifolds of type  $A$  and suppose  $(f_j)$  is a sequence of  $L$ -BLD-mappings  $M \rightarrow N$  converging pointwise to a continuous mapping  $f: M \rightarrow N$ . Then  $f$  is  $L$ -BLD.*

Note that in [MV88] Martio and Väisälä prove a corresponding sharp result in the Euclidean setting, and in [HK11] Heinonen and Keith show that in the setting of the so called generalized manifolds of type  $A$  the limit map is  $K$ -BLD with  $K$  depending only on the data.

*Remark 1.6.* The radially conditions are also connected to quasiregular mappings. For example in [OR09, Definition 4.1] quasiregular mappings

are defined as orientation preserving branched covers for which the function  $H_f(x) := \limsup_{r \rightarrow 0} \frac{L(x, f, r)}{l(x, f, r)}$  is everywhere finite and has bounded essential supremum; see also [GW].

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## 2. PRELIMINARY NOTIONS

A mapping between topological spaces is said to be *open* if the image of every open set is open and *discrete* if the point inverses are discrete sets. A continuous, discrete and open mapping is called a *branched cover*.

The *length*  $\ell(\beta)$  of a path  $\beta: [0, 1] \rightarrow X$  in a metric space is defined as

$$\ell(\beta) := \left\{ \sum_{j=1}^k d(\beta(t_{j-1}), \beta(t_j)) \mid 0 = t_0 \leq \dots \leq t_k = 1 \right\}.$$

Paths with finite and infinite length are called rectifiable and unrectifiable, respectively. A metric space  $(X, d)$  is a *path-metric space* if

$$d(x, y) = \inf \{ \ell(\gamma) \mid \gamma: [0, 1] \rightarrow X, \gamma(0) = x, \gamma(1) = y \}$$

for all  $x, y \in X$ . In a similar vein, a metric space  $(X, d)$  is *(C-)quasiconvex* if for all  $x, y \in X$  there exist a path  $\beta: [0, 1] \rightarrow X$  with  $\beta(0) = x$ ,  $\beta(1) = y$  and  $\ell(\beta) \leq Cd(x, y)$ . A 1-quasiconvex space is called a *geodesic space*. Note that by the Hopf-Rinow theorem (see e.g. [Gro99, Hopf-Rinow theorem, p.9]) complete and locally compact path-metric spaces are geodesic and proper; i.e. closed balls are compact. Geodesic spaces are always locally and globally connected. Throughout this section  $X$  and  $Y$  are locally compact and complete path-metric spaces and  $f: X \rightarrow Y$  a branched cover. Furthermore we denote

$$N(y_0, f, A) := \#(A \cap f^{-1}\{y_0\}) \quad \text{and} \quad N(f, A) := \sup_{y \in A} N(y, f, A),$$

where  $A \subset X$  and  $y_0 \in Y$ .

We follow the conventions of [Ric93] and say that  $U \subset X$  is a *normal domain* (for  $f$ ) if  $U$  is a precompact domain such that  $\partial f(U) = f(\partial U)$ . A normal domain  $U$  is a *normal neighbourhood* of  $x \in U$  if  $\overline{U} \cap f^{-1}(\{f(x)\}) = \{x\}$ . By  $U(x, f, r)$  we denote the component of the open set  $f^{-1}(B_Y(f(x), r))$  containing  $x$ . The existence of arbitrarily small normal neighbourhoods is essential for the theory of branched covers. Heuristically normal domains for branched covers have the same role as completeness has for BLD-mappings. The following lemma guarantees the existence of normal domains and the proof is the same as in [Ric93, Lemma I.4.9, p.19], see also [Väi66, Lemma 5.1.].

**Lemma 2.1.** *Let  $X$  and  $Y$  be locally compact complete path-metric spaces and  $f: X \rightarrow Y$  a branched cover. Then for every point  $x \in X$  there exists a radius  $r_x > 0$  such that  $U(x, f, r)$  is a normal neighbourhood of  $x$  for any  $r \in (0, r_x)$ .*

The following corollary is an immediate consequence of Lemma 2.1 and the precompactness of normal domains.

**Corollary 2.2.** *Let  $f: X \rightarrow Y$  be a branched cover between locally compact complete path-metric spaces and  $U \subset X$  a normal domain. Then for any  $y \in f(U)$  there exists a radius  $r_y > 0$  such that for every  $r \in (0, r_y)$  the domains  $U(z, f, r)$  are disjoint normal neighbourhoods of the points  $z \in U \cap f^{-1}(\{y\})$  with*

$$U \cap f^{-1}(B(y, r)) = \bigcup_{z \in U \cap f^{-1}(\{y\})} U(z, f, r).$$

As noted in the introduction, the definition of Lipschitz quotient mappings is equivalent to a local definition in the setting of complete and locally compact path-metric spaces.

**Lemma 2.3.** *Let  $X$  and  $Y$  be complete and locally compact path-metric spaces. Suppose  $f: X \rightarrow Y$  is locally  $L$ -LQ, i.e. for all  $x \in X$  there exist  $r_0 > 0$  such that  $(LQ)$  holds for all  $0 < r < r_0$ . Then  $f$  is  $L$ -LQ.*

*Proof.* Fix a point  $x_0 \in X$  and denote by  $I$  the set of those  $s \in (0, \infty)$  for which  $(LQ)$  holds at  $x_0$  for all  $r \leq s$ . By our assumption  $I$  is a non-empty interval. Suppose  $I \neq (0, \infty)$ . Then the supremum  $\sup I$  exists and a straightforward calculation shows that  $\sup I \in I$ . Since  $X$  is a proper geodesic space, the set  $\partial B_X(x_0, \sup I)$  is a non-empty compact set. By applying the local  $(LQ)$  condition at all points of the boundary  $\partial B_X(x_0, \sup I)$  we see by the compactness of the boundary that there exists  $\varepsilon > 0$  with  $\sup I + \varepsilon \in I$ , which is a contradiction. Thus  $I = (0, \infty)$  and the claim holds true.  $\square$

As mentioned in the introduction, radial mappings have an equivalent definition which we give next. This is in fact the formulation used by Martio and Väisälä, see [MV88, Corollary 2.13].

**Lemma 2.4.** *A mapping  $f: X \rightarrow Y$  between metric spaces is  $L$ -radial if and only if for any point  $x \in X$  there exists a radius  $r_0 > 0$  such that*

$$(R^\#) \quad L^{-1}d(x, y) \leq d(f(x), f(y)) \leq Ld(x, y)$$

for all  $y \in B_X(x, r_0)$ .

*Proof.* The claim follows immediately as we note that the radius  $r_0$  is the same as the radius in the definition of radial mappings.  $\square$

**2.1. Path-lifting methods.** The main tool in the proof of Theorem 1.1 is *path-lifting*. Given a mapping  $f: X \rightarrow Y$  and a path  $\beta: [0, 1] \rightarrow Y$  we say that a path  $\tilde{\beta}: I \rightarrow X$ , where  $I$  is an interval containing 0, is a *lift* of  $\beta$  if  $f \circ \tilde{\beta} = \beta|_I$ . A lift is called a *maximal lift* if it is not a proper restriction of another lift. Finally a lift is a *total lift* if  $I = [0, 1]$ . The existence of lifts give rise to maximal lifts via a straightforward Zorn's lemma argument, see e.g. [Ric93, Theorem 3.2. p.22].

The following path-lifting theorem of Floyd [Flo50] would be sufficient for the purposes of this paper. Recall that a mapping is *light* if the pre-image of a point is totally disconnected; discrete mappings are always light.

**Theorem (Floyd).** *Suppose  $f: X \rightarrow Y$  is a light mapping between two compact metric spaces. Then  $f$  is open if and only if there exists for any path  $\beta: [0, 1] \rightarrow Y$  and any point  $x \in f^{-1}(\beta(0))$  a total lift of  $\beta$  starting from  $x$ .*

We give, for the reader's convenience, a self-contained proof of a special case (Proposition 2.6) of Floyd's theorem. The idea in the proof of Proposition 2.6 is partly motivated by the proof of the classical application of the Baire category theorem e.g. in [KN01, 2.5.56, p.76]. The core of the proof is a Baire category theorem argument, which we formulate as Lemma 2.5 for the sake of clarity.

Let  $f: X \rightarrow Y$  be a branched cover between locally compact and complete path-metric spaces,  $U_0$  a normal domain for  $f$  and  $\beta: [0, 1] \rightarrow f(U_0)$  a path. For any compact set  $J \subset [0, 1]$  we say that an open interval  $W_J \subset [0, 1]$  intersecting  $J$  is a *lifting interval* for  $J$  if there exists a point  $t_J \in W_J \cap J$ , a positive number  $k_J \in \mathbb{N}$ , and a radius  $r_J > 0$  such that

(LI1)  $\#(U_0 \cap f^{-1}(\beta(t))) = k_J$  for all  $t \in W_J \cap J$ ,

(LI2) for  $\{z_J^1, \dots, z_J^{k_J}\} := U_0 \cap f^{-1}(\beta(t_J))$  we have

$$U_0 \cap f^{-1}(B_Y(\beta(t_J), r_J)) = \bigcup_{i=1}^{k_J} U(z_J^i, f, r_J),$$

where the union on the right hand side is disjoint, and

(LI3) there exists mappings  $g_J^i: W_J \cap J \rightarrow U(z_J^i, f, r_J)$ , for  $i = 1, \dots, k_J$ , for which  $f \circ g_J^i = \beta|_{W_J \cap J}$  and the images of the mappings  $g_J^i$  cover all of  $U_0 \cap f^{-1}(\beta(W_J \cap J))$ .

**Lemma 2.5.** *Let  $f: X \rightarrow Y$  be a branched cover between locally compact and complete path-metric spaces,  $U_0$  a normal domain for  $f$ , and  $\beta: [0, 1] \rightarrow f(U_0)$  a path. For any compact set  $J \subset [0, 1]$ , there exists a lifting interval  $W_J$  of  $J$ .*

*Proof.* For any compact subset  $J \subset [0, 1]$  and  $k \in \mathbb{N}$ , we denote

$$N_J^{\geq k} := \{t \in J \mid \#(U_0 \cap f^{-1}(\{\beta(t)\})) \geq k\},$$

$$N_J^{\leq k} := \{t \in J \mid \#(U_0 \cap f^{-1}(\{\beta(t)\})) \leq k\},$$

and  $N_J^k = N_J^{\geq k} \cap N_J^{\leq k}$ .

Since  $f$  is an open continuous map, the set  $N_J^{\geq k}$  is open. Thus the complementary set  $N_J^{\leq k} = J \setminus N_J^{\geq k+1}$  is closed for all  $k \in \mathbb{N}$ . Since  $f$  is discrete the sets  $\{N_J^{\leq k} \mid k \in \mathbb{N}\}$  form a countable closed cover of the compact set  $J$ . Thus, by the Baire category theorem there exists a minimal index  $k_J \in \mathbb{N}$  for which the set  $N_J^{\leq k_J}$  has interior points in  $J$ . Since  $k_J$  is minimal, also the set  $N_J^{k_J}$  has interior points in  $J$ . This means that there exists an open interval  $V \subset [0, 1]$  with  $V \cap J \subset N_J^{k_J}$ , so for all  $t \in V \cap J$

$$\#(U \cap f^{-1}(\{\beta(t)\})) = k_J.$$

Let  $t_J \in V \cap J$  and denote  $\{z_J^1, \dots, z_J^{k_J}\} := U_0 \cap f^{-1}(\{\beta(t_J)\})$ . Let also  $r_J > 0$  be a radius so small that the sets  $U(z_J^i, f, r_J)$  are disjoint normal neighbourhoods of the points  $z_i$  for  $i = 1, \dots, k_J$  satisfying

$$U_0 \cap f^{-1}(B(\beta(t_J), r_J)) = \bigcup_{i=1}^{k_J} U(z_J^i, f, r_J)$$

as in Corollary 2.2. Set  $W_J \subset V$  to be an open interval around  $t_J$  with  $\beta(W_J) \subset B(\beta(t_J), r_J)$ . Since  $f$  is an open map, the restriction of  $f$  to the pre-image of  $\beta(W_J \cap J)$  in  $U_0$  is locally injective by the definition of  $k_J$ . Thus for any compact set  $K \subset W_J \cap J$  the pre-image  $f^{-1}(\beta(K))$  is compact and as a locally injective map between compact sets in Hausdorff spaces the restriction of  $f$  to  $f^{-1}(\beta(K))$  is a local homeomorphism. This local inverse yields maps

$$g_J^i: W_J \cap J \rightarrow U(z_J^i, f, r)$$

satisfying  $f \circ g_J^i = \beta|_{W_J \cap J}$ , for  $i = 1, \dots, k_J$ . Furthermore the images of these lifts cover all of  $U_0 \cap f^{-1}(\beta(W_J \cap J))$ .  $\square$

**Proposition 2.6.** *Let  $f: X \rightarrow Y$  be a branched cover between locally compact and complete path-metric spaces. Suppose  $U_0$  is a normal domain for  $f$  and let  $\beta: [0, 1] \rightarrow f(U_0)$  be a path. Then for any  $x_0 \in U_0 \cap f^{-1}(\{\beta(0)\})$  there exists a total lift of  $\beta$  starting from  $x$ , i.e. a path  $\tilde{\beta}: [0, 1] \rightarrow U_0$  for which  $\tilde{\beta}(0) = x_0$  and  $f \circ \tilde{\beta} = \beta$ .*

*Proof.* To use Lemma 2.5 to construct lifts of  $\beta$ , let  $\mathcal{I}$  be the collection of all intervals  $(a, b) \subset [0, 1]$  such that for any  $c \in (a, b)$  and any  $x \in U_0 \cap f^{-1}(\{\beta(c)\})$  there exists a path  $\alpha: (a, b) \rightarrow U_0$  with  $\alpha(c) = x$  and  $f \circ \alpha = \beta|_{(a, b)}$ . The definition immediately yields that the collection  $\mathcal{I}$  is closed under restrictions to open subintervals, finite non-empty intersections and finite unions when the union is an interval. Furthermore if  $(a_j, b_j) \in \mathcal{I}$ ,  $j \in S$ , is any collection with a connected union, a straightforward argument shows that also  $(\inf_j a_j, \sup_j b_j) \in \mathcal{I}$ . We conclude that  $\mathcal{I}$  is closed under arbitrary connected unions. The rest of the proof is dedicated into showing first that  $\mathcal{I}$  is non-empty, second that  $\cup \mathcal{I}$  is dense in  $[0, 1]$  and finally that  $(0, 1) \in \mathcal{I}$ .

Applying Lemma 2.5 with  $J$  a closed subinterval of  $[0, 1]$  yields an interval  $W_J \subset [a, b]$  and  $k_J$  lifts  $g_J^1, \dots, g_J^{k_J}$  of  $\beta|_{W_J}$  covering all of the pre-image of  $\beta(W_J)$  in  $U_0$ . From this we conclude that  $\mathcal{I}$  contains  $W_J$  and thus is not

empty. In fact, this argument shows that every closed interval  $J \subset [0, 1]$  contains an open subinterval  $I \subset W_J \cap \text{int } J$  with  $I \in \mathcal{I}$ . Thus  $\cup \mathcal{I}$  is dense in  $[0, 1]$ .

We show next that  $(0, 1) \in \mathcal{I}$ . The collection  $\mathcal{I}$  is closed under connected unions, so it suffices to show that  $\cup \mathcal{I} = (0, 1)$ . Suppose  $\cup \mathcal{I} \neq (0, 1)$ . Since the collection  $\mathcal{I}$  is closed under connected unions, we may write  $\cup \mathcal{I}$  as a countable union of disjoint open intervals  $(a_j, b_j)$  for  $j \in \mathbb{N}$ . We apply Lemma 2.5 to the compact set

$$J := [0, 1] \setminus \bigcup_j (a_j, b_j)$$

and obtain a lifting interval  $W_J$  for  $J$  together with the related points  $z_J^1, \dots, z_J^{k_J}$  as in (LI2). We claim that  $W_J \in \mathcal{I}$ . Let  $c \in W_J$  and fix a pre-image  $z_0 \in U_0 \cap f^{-1}(\{\beta(c)\})$ . Let  $i_0 \in \{1, \dots, k_J\}$  be the unique index for which  $z_0 \in U(z_J^{i_0}, f, r_J)$ . To define a lift  $\gamma: W_J \rightarrow U(z_J^{i_0}, f, r_J) \subset U_0$  of  $\beta|_{W_J}$  we set first  $\gamma|_{W_J \cap J} = g_J^{i_0}$ . Since  $\cup \mathcal{I} = \bigcup_{j \in \mathbb{N}} (a_j, b_j)$  is dense in  $[0, 1]$ , there exists intervals  $(a_j, b_j)$  intersecting  $W_J$ . For each of the intervals  $(a_j, b_j) \subset W_J$  there exists at least one lift  $\alpha_j$  of  $\beta|_{(a_j, b_j)}$  with  $|\alpha_j| \cap U(z_J^{i_0}, f, r_J) \neq \emptyset$ ; if  $c \in (a_j, b_j)$  we take the lift of  $\beta|_{(a_j, b_j)}$  containing  $z_0$ , otherwise we take any one of the finitely many possibilities. Since

$$\beta((a_j, b_j)) \subset \beta(W_J) \subset B(\beta(t_J), r_J)$$

and  $U(z_J^{i_0}, f, r_J)$  is normal domain,  $|\alpha_j| \subset U(z_J^{i_0}, f, r_J)$ . Thus we may set  $\gamma|_{(a_j, b_j)} = \alpha_j$ . For the two possible intervals intersecting  $W_J$  but not contained in  $W_J$  we fix lifts in  $U_0$  in a similar vein by studying the intersection of such intervals with  $W_J$ . From the definition of a lifting interval we immediately see that the lift  $\gamma$  thus defined is continuous. Thus  $W_J \in \mathcal{I}$ , which is a contradiction with the definition of  $J$  and we conclude that  $\cup \mathcal{I} = (0, 1)$ . Since  $\mathcal{I}$  is closed under connected unions,  $(0, 1) \in \mathcal{I}$ .

The fact that  $(0, 1) \in \mathcal{I}$  implies the existence of lifts of  $\beta|_{(0,1)}$ . To conclude the proof we need to show that there exists a lift of the whole path  $\beta$  with  $\tilde{\beta}(0) = x_0$ . Let  $r > 0$  be so small that  $U(x_0, f, r)$  is a normal neighbourhood of  $x_0$ . Take  $c \in (0, 1)$  such that  $\beta|_{(0,c]} \subset B(\beta(0), r)$  and fix  $x \in U(x_0, f, r) \cap f^{-1}(\{\beta(c)\})$ . Since  $(0, 1) \in \mathcal{I}$ , there exists a lift  $\tilde{\beta}: (0, 1) \rightarrow U_0$  of  $\beta|_{(0,1)}$  with  $\tilde{\beta}(c) = x$ . The restriction  $\tilde{\beta}|_{(0,c)}: (0, c) \rightarrow U(x_0, f, r)$  is a lift of  $\beta: (0, c) \rightarrow B(\beta(0), r)$ , and since  $U(x_0, f, r)$  is a normal neighbourhood of  $x_0$ ,  $\lim_{s \rightarrow 0} \tilde{\beta}(s) = x_0$ . Thus the lift  $\tilde{\beta}: (0, 1) \rightarrow U_0$  extends to a lift  $\tilde{\beta}': [0, 1) \rightarrow U_0$  with  $\tilde{\beta}(0) = x_0$ . Since  $U_0$  is a normal neighbourhood and  $f$  is discrete, the limit  $\lim_{s \rightarrow 1} \tilde{\beta}(s)$  will equal one of the finitely many pre-images  $U_0 \cap f^{-1}(\{\beta(1)\})$ . Thus the lift extends to the whole interval  $[0, 1]$  and the claim holds true.  $\square$

*Remark 2.7.* We note that the proofs of Lemma 2.5 and Proposition 2.6 are valid for branched covers between locally compact, locally and globally path-connected Hausdorff spaces.

In the proof of Lemma 2.5 the fact that  $\beta$  is a *path*, i.e. that it is defined on an interval, has no role. Indeed the result holds true when “lifting interval” is replaced by “lifting domain” for any mapping  $\beta: Z \rightarrow f(U_0)$  where  $Z$  is



a Baire space. The proof of the Proposition 2.6 does, on the contrary, rely heavily on the fact that  $\beta$  is defined on an interval.

By the theorem of Floyd and Proposition 2.6 we do not obtain the maximal family of lifts as in Rickman's path lifting theorem [Ric73]. A single total lift of a given path within a normal domain is, however, sufficient for our methods. For rectifiable paths and BLD-mappings the local path-lifting extends to a global lift in the following sense.

**Lemma 2.8.** *Let  $f: X \rightarrow Y$  be an  $L$ -BLD mapping between two locally compact and complete path-metric spaces for  $L \geq 1$ . Then any rectifiable path  $\beta: [0, 1] \rightarrow Y$  has a total lift starting from any point  $x_0 \in f^{-1}(\{\beta(0)\})$ .*

*Proof.* The claim follows immediately from noting that lifts of rectifiable paths are contained in closed balls which are compact in the setting of the lemma.  $\square$

Proposition 2.6 also yields the following corollary.

**Corollary 2.9.** *Let  $f: X \rightarrow Y$  be a branched cover between locally compact complete path-metric spaces. Let  $x \in X$  and let  $U$  be a normal neighbourhood of  $x$ . Then for any connected open set  $W \subset f(U)$  with  $f(x) \in W$  the pre-image  $U \cap f^{-1}W$  is connected.*

*Proof.* Let  $y \in W$  and connect  $f(y)$  to  $f(x)$  with a path  $\alpha: [0, 1] \rightarrow W$ . For any point  $z \in f^{-1}\{y\}$  there exists, by Proposition 2.6, a lift  $\tilde{\alpha}: [0, 1] \rightarrow U$  with  $\alpha(0) = z$ . Thus  $z$  belongs to the same component of  $f^{-1}W$  as  $x$ , which proves the claim.  $\square$

### 3. PROOF OF THEOREM 1.1

In the proof of Theorem 1.1 the most involved part is to show that the radially condition (R) implies the path-length condition (BLD) with the same constant (see also [HM15a]). This implication is true in the setting of general path-metric spaces and does not require  $f$  to be open. Thus we state it as the following separate lemma. We thank Jussi Väisälä for this short and elementary proof.

**Lemma 3.1.** *Let  $f: X \rightarrow Y$  be an  $L$ -radial mapping between path-metric spaces and  $\beta: [0, 1] \rightarrow X$  a path. Then*

$$L^{-1}\ell(\beta) \leq \ell(f \circ \beta) \leq L\ell(\beta).$$

*Proof.* Let  $\beta: [0, 1] \rightarrow X$  be a path. The inequality  $\ell(f \circ \beta) \leq L\ell(\beta)$  immediately follows from the fact that between path-metric spaces an  $L$ -radial mapping is an  $L$ -Lipschitz mapping.

To prove the inequality  $L^{-1}\ell(\beta) \leq \ell(f \circ \beta)$ , let  $K = \{x_0, \dots, x_k\} \subset [0, 1]$  with

$$0 = x_0 < x_1 < \dots < x_k = 1$$

and denote

$$C := \sum_{i=1}^k d(\beta(x_{i-1}), \beta(x_i)).$$

Also let  $A$  be the set of those points  $a \in [0, 1]$  for which there exists a finite set  $J(a) := \{t_0, \dots, t_m\}$  such that

- (1)  $0 = t_0 < \dots < t_m = a$ ,
- (2)  $K \cap [0, a] \subset J(a)$ , and
- (3)  $\sum_{i=1}^m d(\beta(t_{i-1}), \beta(t_i)) \leq L \sum_{i=1}^k d(f(\beta(t_{i-1})), f(\beta(t_i)))$ .

We note that  $0 \in A$ , so  $A$  is not empty. Thus the supremum of  $A$  exists and by applying the radially condition at the point  $f(\beta(\sup A))$  we observe first that  $\sup A \in A$  and then that  $\sup A = 1$ . Now for  $J(1) =: \{t_0, \dots, t_N\}$  we have by the triangle inequality that

$$\begin{aligned} C &= \sum_{i=1}^k d(\beta(x_{i-1}), \beta(x_i)) \leq \sum_{i=1}^N d(\beta(t_{i-1}), \beta(t_i)) \\ &\leq L \sum_{i=1}^N d(f(\beta(t_{i-1})), f(\beta(t_i))) \leq L\ell(f \circ \beta). \end{aligned}$$

Thus we have that  $\ell(\beta) \leq L\ell(f \circ \beta)$ .  $\square$

*Proof of Theorem 1.1.* We note that under any of the conditions (i)–(iv), the mapping  $f$  is an  $L$ -Lipschitz branched cover. We prove the theorem by two sequences of implications, showing first  $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$  and then completing the equivalence by showing  $(i) \Rightarrow (iv) \Rightarrow (iii)$ .

The proof of the implication  $(i) \Rightarrow (ii)$  is essentially from [HR02, Proposition 4.13 and Remark 4.16(c)]. To show that an  $L$ -BLD-mapping is an  $L$ -LQ mapping it suffices to show that the co-Lipschitz condition  $B_Y(f(x), L^{-1}r) \subset f(B_X(x, r))$  holds, since the other inclusion is equivalent to the mapping being  $L$ -Lipschitz. Since  $Y$  is a proper space, the bounded set  $f(B_X(x, r))$  is precompact. Thus we may fix  $z_0 \in \partial f(B_X(x, r))$  with

$$d(f(x), z_0) = d(f(x), \partial f(B_X(x, r))),$$

and a geodesic  $\beta: [0, 1] \rightarrow Y$  with  $\beta(0) = f(x)$  and  $\beta(1) = z_0$ . By Lemma 2.8 there exists a total lift  $\alpha$  of this path starting from  $x$ . On the other hand, since  $f$  is an open map and

$$\alpha(1) \in f^{-1}(\{z_0\}) \subset f^{-1}(\partial f(B_X(x, r))),$$

we have  $\alpha(1) \in \partial B_X(x, r)$ . Thus

$$\ell(\alpha) \geq d(\alpha(0), \alpha(1)) \geq d(x, \partial B_X(x, r)) = r.$$

By combining this with (BLD) and the fact that  $\beta$  is a geodesic

$$d(f(x), \partial f(B_X(x, r))) = d(f(x), z_0) = \ell(\beta) \geq L^{-1}\ell(\alpha) \geq L^{-1}r.$$

Thus  $B_Y(f(x), L^{-1}r) \subset f(B_X(x, r))$ .

Suppose next that  $f$  is a discrete  $L$ -LQ-mapping. Let  $x \in X$ . Since  $f$  is discrete, there is a positive distance  $r_0 := d(x, f^{-1}(\{f(x)\}) \setminus \{x\})$ . We claim that for any  $r < r_0/2$  we have

$$d(f(x), (f\partial B_X(x, r))) \geq L^{-1}r.$$

To see this let  $z \in \partial B_X(x, r)$  and note that  $B_X(z, r) \cap f^{-1}(\{f(x)\}) = \emptyset$ , so  $f(x) \notin f(B_X(z, r))$ . Since  $f(B_X(z, r))$  contains the ball  $B_Y(f(z), L^{-1}r)$ ,

we have  $d(f(z), f(x)) \geq L^{-1}r$ . Thus  $f$  is  $L$ -radial at  $x$ , since the condition  $L(x, f, r) \leq Lr$  is equivalent to the  $L$ -Lipschitz condition.

By Lemma 3.1 an  $L$ -radial branched cover is  $L$ -BLD. Thus we have shown that the conditions (i), (ii) and (iii) are equivalent.

The proof that an  $L$ -BLD mapping is  $L$ -coradial, let  $x \in X$  and fix  $r_x$  as in Lemma 2.1. Denote

$$r_0 := \min\{r_x, d(x, f^{-1}(\{f(x)\}) \setminus \{x\}) / (2L)\}$$

and fix  $r < r_0$ . Let  $z \in \partial B_Y(f(x), r)$  and let  $\beta_z: [0, 1] \rightarrow \overline{B}_Y(f(x), r)$  be a geodesic with  $\beta_z(0) = f(x)$  and  $\beta_z(1) = z$ . By Lemma 2.8 for any  $w \in U(x, f, r) \cap f^{-1}(\{z\})$  there exists a lift  $\tilde{\beta}_z$  of  $\beta_z$  with  $\tilde{\beta}_z(0) = x$  and  $\tilde{\beta}_z(1) = w$ . Since  $f$  is  $L$ -BLD,  $\ell(\tilde{\beta}_z) \leq Lr$ . Thus for all  $r < r_0$  we have  $L^*(x, f, r) \leq Lr$ . Since the condition  $l^*(x, f, r) \geq L^{-1}r$  is equivalent to the  $L$ -Lipschitz condition,  $f$  is  $L$ -coradial at  $x$ .

Suppose finally that  $f$  is a branched cover which satisfies the co-radiality condition (R\*) with constant  $L \geq 1$ . Fix  $x \in X$  and let  $r_0 > 0$  be such that for all  $r < r_0$  the normal neighbourhoods  $U(x, f, r)$  satisfy (R\*). By Corollary 2.9 we have for all  $r < r_0$  that

$$U(x, f, r) = U(x, f, r_0) \cap f^{-1}(B_Y(f(x), r)).$$

Then for each point  $z \in U(x, f, r_0)$  there exists a radius  $r_z > 0$  such that  $z \in \partial U(x, f, r_z)$ . The condition (R\*) then implies that  $d(z, x) \in [L^{-1}r, Lr]$ , so the mapping  $f$  is  $L$ -radial.  $\square$

#### 4. LIMIT THEOREMS FOR BLD-MAPPINGS

To show that the pointwise limit of  $L$ -LQ mappings  $f_j: X \rightarrow Y$  between proper metric spaces is an  $L$ -LQ mapping is a straightforward calculation. In this section we show that a similar limit result holds in the setting of sequences of pointed mapping packages.

A *pointed mapping package* is a triple  $((X, x_0), (Y, y_0), f)$  where  $X$  and  $Y$  are locally compact and complete path-metric spaces having fixed base-points  $x_0 \in X$ ,  $y_0 \in Y$ , and  $f: X \rightarrow Y$  is a continuous mapping satisfying  $f(x_0) = y_0$ . We define the convergence of a sequence of pointed mapping packages as in [DS97, Definition 8.18 and Lemma 8.19], see also [KM, Definition 3.8] and [Dav15, Definition 2.1]. For  $A \subset X$  we denote  $N_\varepsilon(A) := B_X(A, \varepsilon)$ . A map  $\phi: (X, x_0) \rightarrow (Y, y_0)$  between pointed metric spaces is called an  $\varepsilon$ -*quasi-isometry* if

- (i) for all  $a, b \in B_X(x_0, \varepsilon^{-1})$  we have  $|d(\phi(a), \phi(b)) - d(a, b)| < \varepsilon$ , and
- (ii) for all  $\varepsilon \leq r \leq \varepsilon^{-1}$  we have  $N_\varepsilon(\phi(B_X(x_0, r))) \supset B_Y(y_0, r - \varepsilon)$ .

**Definition 4.1.** *Pointed mapping packages  $((X_j, x_j), (Y_j, y_j), f_j)$  for  $j \in \mathbb{N}$  converge to  $((X, x_0), (Y, y_0), f)$  if the following conditions hold:*

- (GH-i) For every  $r > 0$  and  $i \in \mathbb{N}$  there exists  $\varepsilon_i^{(r)} > 0$  and  $\varepsilon_i^{(r)}$ -quasi-isometries

$$\begin{aligned} g_i^{(r)}: B_{X_i}(x_i, r) &\rightarrow N_{\varepsilon_i^{(r)}}\left(g_i^{(r)}(B_{X_i}(x_i, r))\right) \subset X \quad \text{and} \\ h_i^{(r)}: B_{Y_i}(y_i, r) &\rightarrow N_{\varepsilon_i^{(r)}}\left(h_i^{(r)}(B_{Y_i}(y_i, r))\right) \subset Y \end{aligned}$$

so that  $\varepsilon_i^{(r)} \rightarrow 0$  and for all  $i \in \mathbb{N}$ ,  $g_i^{(r)}(x_i) = x_0$ ,  $h_i^{(r)}(y_i) = y_0$ ,

$$B(x_0, r - \varepsilon_i^{(r)}) \subset N_{\varepsilon_i^{(r)}} \left( g_i^{(r)}(B_{X_i}(x_i, r)) \right) \quad \text{and} \\ B(y_0, r - \varepsilon_i^{(r)}) \subset N_{\varepsilon_i^{(r)}} \left( h_i^{(r)}(B_{Y_i}(y_i, r)) \right).$$

(GH-ii) For any  $x \in X$  and all  $r > d(x, x_0)$  we have  $h_i^{(r)}(f_i(a_i)) \rightarrow f(a)$  as  $i \rightarrow \infty$  whenever  $a_i \in X_i$  is a sequence of points with  $g_i^{(r)}(a_i) \rightarrow a$  as  $i \rightarrow \infty$ .

Note that the condition (GH-i) in Definition 4.1 is equivalent to saying that  $(X_j, x_j) \rightarrow (X, x_0)$  and  $(Y_j, y_j) \rightarrow (Y, y_0)$  in the Gromov-Hausdorff sense, see e.g. [DS97]. For fixed spaces, the condition (GH-ii) is just the pointwise convergence of mappings.

**Lemma 4.2.** *Let  $((X_j, x_j), (Y_j, y_j), f_j)$  be a sequence of mapping packages converging to a mapping package  $((X, x_0), (Y, y_0), f)$ . If all the mappings  $f_j$  are  $L$ -LQ, then so is  $f$ .*

*Proof.* We show first that the limiting map  $f$  is  $L$ -Lipschitz. Let  $a, b \in X$  and fix a radius  $R > 0$  with  $a, b \in B_X(x_0, R)$  and  $f(a), f(b) \in B_Y(y_0, R)$ . Fix two sequences of points

$$a_i \in (g_i^{(R)})^{-1} \left( B_X(a, \varepsilon_i^{(R)}) \right) \subset X_i \quad \text{and} \quad b_i \in (g_i^{(R)})^{-1} \left( B_Y(b, \varepsilon_i^{(R)}) \right) \subset X_i.$$

By (GH-ii) we have  $h_i^{(R)}(f_i(a_i)) \rightarrow f(a)$  and  $h_i^{(R)}(f_i(b_i)) \rightarrow f(b)$ , so since each  $f_i$  is  $L$ -Lipschitz the triangle inequality yields

$$d_Y(f(a), f(b)) \leq L d_X(a, b) + 2\varepsilon_i^{(R)}$$

for all  $i \in \mathbb{N}$ . Thus  $f$  is  $L$ -Lipschitz.

To prove the claim it now suffices to show that  $B_Y(f(x), r/L) \subset f(B_X(x, r))$  for each  $x \in X$  and  $r > 0$ . Let  $z_0 \in B_Y(f(x), r/L)$ . Fix a radius  $r_0 < r$  such that  $z_0 \in B_Y(f(x), r_0/L)$ . Let also  $R = 2L(d(x_0, x) + r)$  and let  $(\varepsilon_i^{(R)})$ ,  $(g_i^{(R)})$  and  $(h_i^{(R)})$  be as in Definition 4.1. For each  $i \in \mathbb{N}$  we take

$$c_i \in (g_i^{(R)})^{-1} \left( B_X(x, \varepsilon_i^{(R)}) \right) \subset X_i.$$

By (GH-ii) we have  $h_i(f_i(c_i)) \rightarrow f(x)$ , so  $\delta_i := d_Y(h_i(f_i(c_i)), f(x)) \rightarrow 0$  as  $i \rightarrow \infty$ . Likewise we fix for all  $i \in \mathbb{N}$  a point

$$z_i \in (h_i^{(R)})^{-1} \left( B_Y(z_0, \varepsilon_i^{(R)}) \right) \subset Y_i.$$

Now the triangle inequality yields  $d_{Y_i}(f(c_i), z_i) \leq r_0/L + (\delta_i + 2\varepsilon_i^{(R)})$ , since  $h_i$  is an  $\varepsilon_i^{(R)}$ -quasi-isometry.

Since  $f_i$  is  $L$ -LQ, we have

$$z_i \in B_{Y_i}(f_i(c_i), r_0/L + (\delta_i + 2\varepsilon_i^{(R)})) \subset f \left( B_{X_i}(c_i, r_0 + L(\delta_i + 2\varepsilon_i^{(R)})) \right).$$

Thus there exists, for each  $i \in \mathbb{N}$ , a point

$$a_i \in B_{X_i}(c_i, r_0 + L(\delta_i + 2\varepsilon_i^{(R)})) \cap f_i^{-1}(\{z_i\}).$$

Since  $B_X(x_0, R_0)$  is precompact, we can pass to subsequences  $(\varepsilon_j^{(R)})$ ,  $(g_j^{(R)})$  and  $(h_j^{(R)})$  and assume that the sequence  $(g_j^{(R)}(a_j))$  in  $X$  converges to a point in  $\overline{B}_X(x_0, R_0)$ . Finally we note that since each  $g_j^{(R)}$  is an  $\varepsilon_j^{(R)}$ -quasi-isometry, we have

$$g_j^{(R)}(a_j) \in B_X(x, r_0 + L(\delta_j + 2\varepsilon_j^{(R)}) + 2\varepsilon_j^{(R)}),$$

so  $a \in \overline{B}_X(x, r_0)$ . Thus by (GH-ii),  $f(a) = z$ . This proves that we have

$$z \in f(\overline{B}_X(x, r_0)) \subset f(B_X(x, r)).$$

Since  $z$  was arbitrary,  $B_Y(f(x), L^{-1}r) \subset f(B_X(x, r))$  and  $f$  is  $L$ -LQ.  $\square$

Lemma 4.2 yields immediately the proof of Theorem 1.2 when combined with our characterization Theorem 1.1.

*Proof of Theorem 1.2.* By the characterization Theorem 1.1 the classes of  $L$ -BLD mappings and discrete  $L$ -LQ mappings equal. By Lemma 4.2  $f$  is  $L$ -LQ, and thus  $L$ -BLD.  $\square$

*Remark 4.3.* Also the ultralimit of a sequence of  $L$ -LQ mappings is  $L$ -LQ; see [LP14, Lemma 3.1]. Since completeness, local compactness and a path-metric pass to ultralimits, see e.g. [Kap09], results of this section hold also when the convergence of mapping packages is replaced by an ultralimit. Likewise Theorem 1.4 has a corresponding ultralimit version. The proof is similar in the ultralimit setting as is given here for pointed mapping package convergence.

The limit of a sequence of  $L$ -BLD mappings need not be discrete in general; for example a sequence of 1-BLD-mappings  $S^1 \rightarrow S^1(\frac{1}{k})$ ,  $z \mapsto \frac{1}{k}z^k$ , converges to a constant map  $S^1 \rightarrow \{0\}$ . In what follows we consider the setting of generalized manifolds of type  $A$ , where the existence of certain uniform local multiplicity bounds enable us to show that the limit of  $L$ -BLD mappings is always discrete.

**4.1. Generalized manifolds.** Throughout this section we assume that  $M$  and  $N$  are generalized  $n$ -manifolds having a complete path-metrics. For the definition of generalized manifolds and their basic theory we refer to [HR02]. Note that generalized manifolds are locally compact.

The following lemma is an elementary observation in the local value distribution theory of BLD-mappings. We assume it is known to the specialists in the field but have not found it recorded in the literature.

**Lemma 4.4.** *Let  $f: M \rightarrow N$  be an  $L$ -BLD-mapping between two generalized manifolds equipped with a complete path-metric. Then for any two points  $x, y \in N \setminus f(B_f)$  there exists a bijection  $\psi_f: f^{-1}(\{x\}) \rightarrow f^{-1}(\{y\})$  satisfying  $d_M(a, \psi_f(a)) \leq Ld_N(x, y)$  for all  $a \in f^{-1}(\{x\})$ .*

*Proof.* The claim follows immediately by connecting  $x$  and  $y$  with a geodesic  $\beta$  and taking the maximal sequence of  $f$  liftings of  $\beta$ , see [Ric93]. Indeed, to define the bijection  $\psi_f$ , let  $\beta: [0, 1] \rightarrow N$  be a geodesic with  $\beta(0) = x$  and  $\beta(1) = y$ . For each  $a \in f^{-1}(\{x\})$  the path  $\beta$  has exactly one lift  $\alpha_a$  in the maximal sequence of path liftings of  $\beta$  under  $f$  starting from  $a$ . By Lemma

2.8 maximal lifts of  $\beta$  are total, and we define  $\psi_f(a) = \alpha_a(1) \in f^{-1}(\{y\})$ . Since  $y \in N \setminus f(B_f)$  the mapping  $\psi_f$  is injective, so by symmetry the mapping  $\psi_f$  is bijective. Since  $\beta$  is a geodesic,  $d_X(a, \psi_f(a)) \leq Ld_Y(x, y)$  by (BLD).  $\square$

In the setting of generalized manifolds the multiplicity bounds of Heinonen and Rickman [HR02] are at our disposal for the proof of Theorem 1.4. Note that the proof of the following theorem shows that the limiting map is discrete in a quantitative sense that  $f$  has the same uniform local multiplicity bounds as the mappings in the sequence  $(f_i)$ . This result generalizes the authors previous result [Lui, Theorem 1.2] and the proof here is similar.

*Proof of Theorem 1.4.* By Lemma 4.2 we know that the limiting map is L-LQ. Thus it suffices, by Theorem 1.1, to show that  $f$  is discrete. We show that for each ball  $B_M(x_0, r_0)$  we have  $N(f(x_0), f, B_M(x_0, r_0)) < \infty$ .

By [HR02, Theorem 6.8], we have for any  $j \in \mathbb{N}$ , any  $x \in M$  and any radius  $r > 0$

$$(N) \quad N(f_j(x), f_j, B_{M_j}(x, r)) \leq (Lc_M)^n \frac{\mathcal{H}^n(B_{M_j}(x, \lambda r))}{\mathcal{H}^n(B_{N_j}(f_j(x), (\lambda - 1)r/Lc_N))}$$

for any  $\lambda > 1$ , where  $\mathcal{H}^n$  is the Hausdorff  $n$ -measure and  $c_M$  and  $c_N$  are the quasi-convexity constants of the spaces  $M$  and  $N$ , respectively. We fix  $\lambda = 2$  and note that  $c_M = c_N = 1$  for path-metric spaces. Since we assumed the spaces  $M_j$  and  $N_j$  to have uniform constants, we have for any radius  $r$  a constant  $N_0(r)$  depending only on  $r$  such that

$$(4.1) \quad N(f_j(x), f_j, B_{M_j}(x, r)) \leq N_0(r)$$

for any  $j \in \mathbb{N}$  and  $x \in M_j$ .

Let  $x_0 \in M$  and fix a radius  $r_0 > 0$ . We show that

$$N(f(x_0), f, B_M(x_0, r_0)) \leq N_0(r_0) =: N_0.$$

Suppose towards contradiction that

$$N(f(x_0), f, B_M(x_0, r_0)) \geq N_0 + 1.$$

Then there exists  $y_0 \in f(B_M(x_0, r_0))$  having at least  $N_0 + 1$  preimages in  $B_M(x_0, r_0)$ , that is,

$$B_M(x_0, r_0) \cap f^{-1}(\{y_0\}) \supset \{a_0, \dots, a_{N_0}\}$$

with  $a_i$  mutually disjoint. Denote

$$\delta := \min_{i=0, \dots, N_0} \{d_M(a_i, \{a_k\}_{k \neq i}), d_M(a_i, \partial B_M(x_0, r_0))\}$$

and note that the balls  $B_M(a_i, \delta/2)$  are disjoint.

Take  $R_0 = 2r_0$  and let  $(\varepsilon_j^{(R_0)})$ ,  $(g_j^{(R_0)})$  and  $(h_j^{(R_0)})$  be the sequences associated to  $R_0$  in (GH-i). For each  $j \in \mathbb{N}$  and all  $i = 0, \dots, N_0$  we fix points

$$c_i^j \in (g_j^{(R_0)})^{-1}(B_M(a_i, \varepsilon_j)) \subset M_j.$$

Since the mappings  $g_j^{(R_0)}$  are  $\varepsilon_j^{(R_0)}$ -quasi-isometries, the balls  $B_{M_j}(c_i^j, \delta/4)$  will be disjoint in  $M_j$  for  $\varepsilon_j^{(R_0)} < \delta/4$ . By (GH-ii) we have, for all  $i = 0, \dots, N_0$ , that  $h_j^{(R_0)}(f_j(c_i^j)) \rightarrow f(a_i) = y_0$  as  $i \rightarrow \infty$ . Denote

$$\delta_i := d_N(h_j^{(R_0)}(f_j(c_i^j)), y_0).$$

Since the mappings  $h_j^{(R_0)}$  are  $\varepsilon_j^{(R_0)}$ -quasi-isometries, the triangle inequality yields that

$$d_{N_j}(f_j(c_i^j), f_j(c_k^j)) \leq 4\varepsilon_j^{(R_0)} + \delta_i$$

for all  $i, k \in \{0, \dots, N_0\}$ . Thus, for  $\varepsilon_j^{(R_0)} < \delta/(24L)$ , there exists a point

$$z_0 \in \bigcap_{k=0}^{N_0} B_{N_j}(f(c_k^j), \delta/(4L)) \subset \bigcap_{k=0}^{N_0} fB_{M_j}(c_k^j, \delta/4).$$

Since the balls  $B_{M_j}(c_i^j, \delta/4)$  are disjoint we have

$$\# \left( B_{M_j}(x_0, r_0) \cap f_j^{-1}(\{z_0\}) \right) \geq N_0 + 1.$$

This is a contradiction with (4.1), so  $N(f(x_0), f, r_0) \leq N_0$  and  $f$  is discrete.  $\square$

*Remark 4.5.* The multiplicity bound (N) of Heinonen and Rickman holds more generally in the setting of Ahlfors  $Q$ -regular generalized manifolds equipped with a complete path-metric if  $\mathcal{H}^Q(B_f) = 0$ . Note, however, that this is a very strong assumption; indeed the Heinonen-Rickman conjecture [HR02, Theorem 6.4] asks whether the branch set of a BLD-mapping  $f: M \rightarrow N$  has zero measure in any setting where  $M$  and  $N$  are not quasi-convex generalized  $n$ -manifolds of type A.

Under the assumption  $\mathcal{H}^Q(B_f) = 0$ , the bound (N) follows from standard covering arguments combined with the co-radiality condition (R\*) and Lemma 4.4. Thus also the proof of Theorem 1.4 goes through in this setting.

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